

# Characteristics of Polynomial Functions

## Exercises

1. Consider the following polynomial functions.

a.  $y = -2x^3 + 4x - 5$

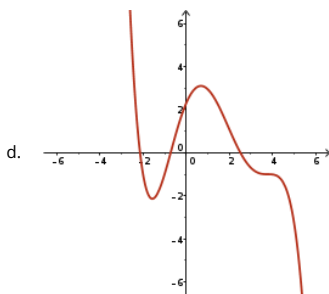
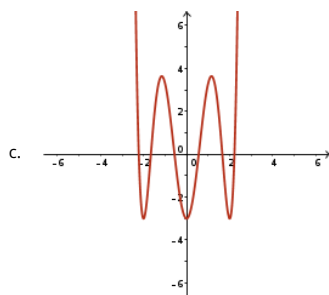
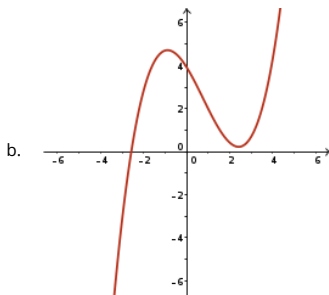
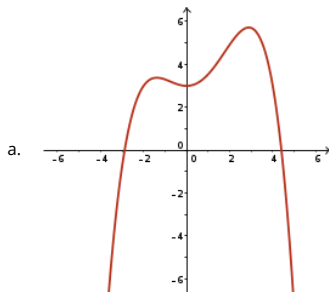
b.  $f(x) = 5x^4 + 2x^3 - 4x^2 + x - 7$

c.  $g(x) = x^5 + 2x^3 - 5x + 8$

For each one, perform the following tasks.

- Describe the end behaviour of the function.
- Determine the maximum and minimum number of turning points.
- Determine the maximum and minimum number of  $x$ -intercepts.

2. Given the graph of the polynomial function  $y = f(x)$ , identify the minimum possible degree of the function and the sign of the leading coefficient.



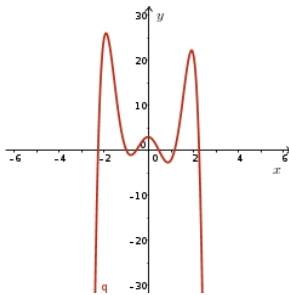
3. Determine whether the following statements are true or false. If false, explain or provide a counter-example.

- a. An even degree polynomial must have an even number of turning points.
- b. An odd degree polynomial must have at least one  $x$ -intercept.
- c. An even degree polynomial function has an absolute maximum or absolute minimum value.
- d. A polynomial function with three  $x$ -intercepts must be of degree three or greater.
- e. A polynomial function with  $n$  turning points must be of degree  $n$ ,  $n \in \mathbb{N}$ .
- f. An odd degree polynomial function can have even symmetry (symmetry about the  $y$ -axis).

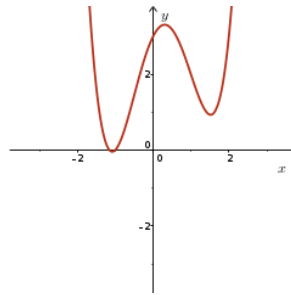
4. Match the equations of the polynomial functions to the most appropriate graph. Justify your choice.

- a.  $y = -x^3 + 2x^2 + 2x + 3$
- b.  $y = x^4 - x^3 - 3x^2 + 2x + 3$
- c.  $y = x^4 + 5x^3 + 6x^2 - 4x - 3$
- d.  $y = x^3 - 3x^2 + 5x - 3$
- e.  $y = -2x^6 + 13x^4 - 15x^2 - x + 3$
- f.  $y = -2x^5 - 7x^4 + 3x^3 + 18x^2 + 3$

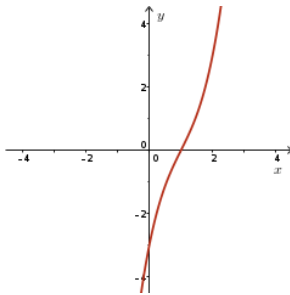
I.



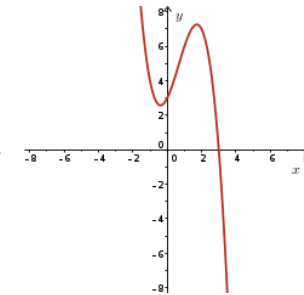
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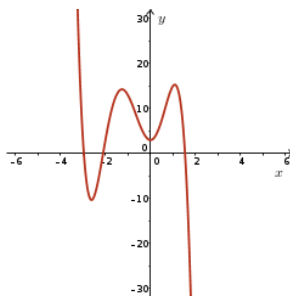
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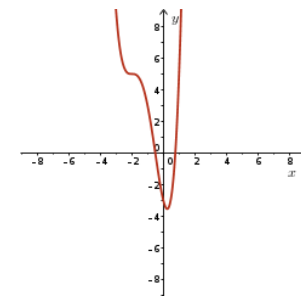
IV.



V.



VI.



5. What possible number of zeros can a quartic polynomial function have? Sketch a quartic polynomial function with a negative leading coefficient for each possibility?

6. Sketch a graph of a polynomial function that satisfies each set of conditions.

- a. Degree three, two distinct  $x$ -intercepts, two turning points, and end behaviour such that  $y \rightarrow \infty$  as  $x \rightarrow -\infty$  and  $y \rightarrow -\infty$  as  $x \rightarrow \infty$
- b. Degree four, two distinct  $x$ -intercepts, three turning points, and end behaviour such that  $y \rightarrow \infty$  as  $x \rightarrow \pm\infty$
- c. Degree four, negative leading coefficient, three distinct  $x$ -intercepts, three turning points
- d. Degree three, positive leading coefficient, one  $x$ -intercept, two turning points
- e. Degree five, negative leading coefficient, two distinct  $x$ -intercepts, two turning points
- f. Degree five, positive leading coefficient, one  $x$ -intercept, four turning points

7. Sketch a possible graph of each function by identifying the end behaviours and determining the  $x$ - and  $y$ -intercepts of the function.

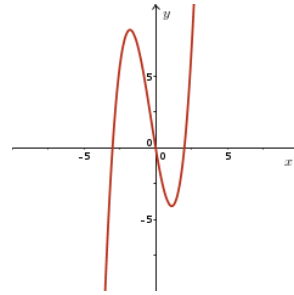
a.  $f(x) = (x - 1)(x - 3)(x + 1)(x + 4)$

b.  $y = -2x^3 - 3x^2 + 9x$

8. Given the graph of the polynomial function  $f(x) = x^3 + x^2 - 6x$ , sketch the graph of

a.  $y = |f(x)|$

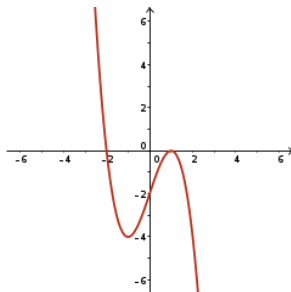
b.  $y = f(|x|)$



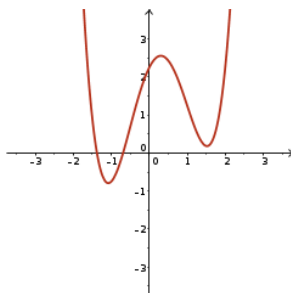
# Characteristics of Polynomial Functions

## Partial Solutions

1. There is no solution provided for this question.
2.
  - a. Since there are three turning points, the function must have an even degree of at least 4. Therefore the function's minimum degree is 4. By inspection of the end behaviours of the graph, the function must have a negative leading coefficient.
  - b. Since there are two turning points, the function must have an odd degree of at least 3. Therefore the function's minimum degree is 3. By inspection of the end behaviours of the graph, the function must have a positive leading coefficient.
  - c. Since there are five turning points, the function must have an even degree of at least 6. Therefore the function's minimum degree is 6. By inspection of the end behaviours of the graph, the function must have a positive leading coefficient.
  - d. Since there are two turning points, the function must have an odd degree of at least 3. But it also has a point of inflection that is not between the two turning points, so it must be degree at least 5. Therefore, the function's minimum degree is 5. By inspection of the end behaviours of the graph, the function must have a negative leading coefficient.
3. There is no solution provided for this question.
4.
  - a. The equation is a cubic with a negative leading coefficient, so it has at most 2 turning points, and  $y \rightarrow \infty$  as  $x \rightarrow -\infty$  and  $y \rightarrow -\infty$  as  $x \rightarrow \infty$ . The only graph which satisfies these conditions is IV.
  - b. The equation is a quartic with a positive leading coefficient, so it has at most 3 turning points, and  $y \rightarrow \infty$  as  $x \rightarrow \pm\infty$ . Note that  $f(0) = 3$ , so it has a y-intercept of 3. The only graph which satisfies these conditions is II.
  - c. The equation is a quartic with a positive leading coefficient, so it has at most 3 turning points and  $y \rightarrow \infty$  as  $x \rightarrow \pm\infty$ . Note that  $f(0) = -3$ , so it has a y-intercept of  $-3$ . The only graph which satisfies these conditions is VI.
  - d. The equation is a cubic with a positive leading coefficient, so it has at most 2 turning points, and  $y \rightarrow -\infty$  as  $x \rightarrow -\infty$  and  $y \rightarrow \infty$  as  $x \rightarrow \infty$ . The only graph which satisfies these conditions is III.
  - e. The equation is a sixth degree function with a negative leading coefficient, so it has at most 5 turning points and  $y \rightarrow \infty$  as  $x \rightarrow \pm\infty$ . The only graph which satisfies these conditions is I.
  - f. The equation is a fifth degree function with a negative leading coefficient, so it has at most 4 turning points and  $y \rightarrow \infty$  as  $x \rightarrow -\infty$ ,  $y \rightarrow -\infty$  as  $x \rightarrow \infty$ . V satisfies these conditions, and cannot be cubic since it has 4 turning points. Therefore, V is the graph of this equation.
5. There is no solution provided for this question.
6.
  - a. The graph will start in the 2<sup>nd</sup> quadrant and end in the 4<sup>th</sup> quadrant based on the description of the end behaviour. Position the graph of a cubic function, with this end behaviour and three turning points, on the  $x$ - $y$  plane in such a way that the graph intersects the  $x$ -axis at two distinct locations. For this to happen, a turning point must occur at one of the  $x$ -intercepts. A possible solution is

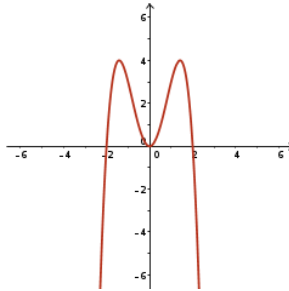


- b. The graph will start in the 2<sup>nd</sup> quadrant and end in the 1<sup>st</sup> quadrant based on the description of the end behaviour. Position the graph of a quartic function, with this end behaviour and three turning points, on the  $x$ - $y$  plane in such a way that the graph intersects the  $x$ -axis at two distinct locations. A possible solution is

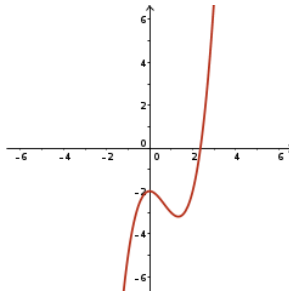


- c. The leading coefficient of the quartic function is negative so the graph will start in the 3<sup>rd</sup> quadrant and end in the 4<sup>th</sup> quadrant. Position the graph of a quartic function, with this end behaviour and three turning points, on the  $x$ - $y$  plane in such a way that the graph intersects

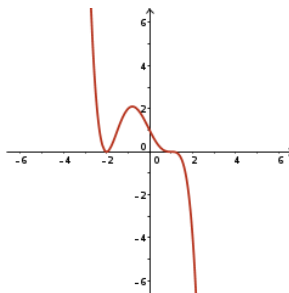
the  $x$ -axis at three distinct locations. For this to happen, a turning point must occur at one of the  $x$ -intercepts. A possible solution is



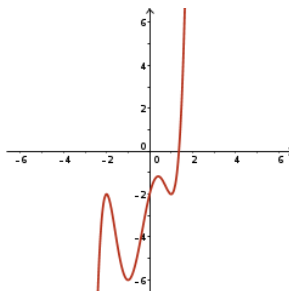
- d. The leading coefficient of the cubic function is positive so the graph will begin in the 3<sup>rd</sup> quadrant and end in the 1<sup>st</sup> quadrant. Position the graph of a cubic function, with this end behaviour and two turning points, on the  $x$ - $y$  plane, in such a way that the graph intersects the  $x$ -axis at exactly one location. For this to happen, both turning points must occur off of the  $x$ -axis and are either both above or below the  $x$ -axis. A possible solution is



- e. The leading coefficient of the quintic function is negative so the graph will begin in the 2<sup>nd</sup> quadrant and end in the 4<sup>th</sup> quadrant. Position the graph of a quintic function, with this end behaviour and two turning points, on the  $x$ - $y$  plane, in such a way that the graph intersects the  $x$ -axis at two distinct locations. For this to happen, a turning point must occur at one of the  $x$ -intercepts. A possible solution is



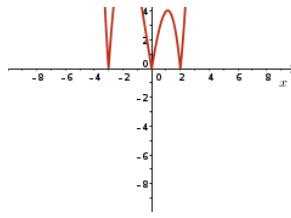
- f. The leading coefficient of the quintic function is positive so the graph will begin in the 3<sup>rd</sup> quadrant and end in the 1<sup>st</sup> quadrant. Position the graph of a quintic function, with this end behaviour and four turning points, on the  $x$ - $y$  plane, in such a way that the graph intersects the  $x$ -axis at exactly one location. A possible solution is



7. There is no solution provided for this question.

8. a. Any point  $(x, y)$  on  $y = f(x)$  will map to the point  $(x, |y|)$  on  $y = |f(x)|$ . That is, the graph of  $y = |f(x)|$  is the graph of  $y = f(x)$  with the portions that lie below the  $x$ -axis reflected vertically about the  $x$ -axis. This gives the sketch





b.  $f(|x|) = |x|^3 + |x|^2 - 6|x|$ . Thus  $f(-k) = f(k)$  for all  $k \geq 0, k \in \mathbb{R}$ . The graph will be symmetrical about the y-axis. To obtain the graph, use the graph of  $y = f(x), x \geq 0$ , and then reflect this branch in the y-axis to complete the graph.

